6 Réponses

Solution de l'exercice 1

$I = \left[x^2\right]_0^1 = 1$; $J = [\sin x]_{\pi/6}^{\pi/3} = \dfrac{\sqrt{3}-1}{2}$; $K = [\ln|x|]_1^e = 1$; $L = [\arctan x]_0^1 = \dfrac{\pi}{4}$

Solution de l'exercice 2

1. $\displaystyle\int_0^1 \frac{2x^3+2x+1}{x^2+1}\,dx = \int_0^1 \left(2x + \frac{1}{x^2+1}\right)\,dx = \left[x^2 + \arctan x\right]_0^1 = 1 + \frac{\pi}{4}$
2. $\displaystyle\int_0^1 \frac{x}{1+x}\,dx = \int_0^1 \left(1 - \frac{1}{1+x}\right)\,dx = [x - \ln|1+x|]_0^1 = 1 - \ln 2$
3. $\displaystyle\int_0^1 \frac{1-4x}{1+2x}\,dx = \int_0^1 \left(-2 + \frac{3}{1+2x}\right)\,dx = \left[-2x + \frac{3}{2}\ln|1+2x|\right]_0^1 = \frac{3}{2}\ln 3 - 2$
4. $\displaystyle\int_0^1 \frac{4x^2}{2x+1}\,dx = \int_0^1 \left(2x - 1 + \frac{1}{2x+1}\right)\,dx = \left[x^2 - x + \frac{1}{2}\ln|2x+1|\right]_0^1 = \frac{1}{2}\ln 3$
5. $\displaystyle\int_4^7 \frac{x+3}{x-3}\,dx = \int_4^7 \left(1 + \frac{6}{x-3}\right)\,dx = [x + 6\ln|x-3|]_4^7 = 6\ln 4 + 3$

Solution de l'exercice 3

1. $\displaystyle\int_0^1 \left(x + 1 + \frac{1}{x+1}\right)\,dx = \left[\frac{1}{2}x^2 + x + \ln|x+1|\right]_0^1 = \frac{3}{2} + \ln 2$
2. $\displaystyle\int_0^{\pi/6} 2\cos 3x\,dx = \left[\frac{2}{3}\sin 3x\right]_0^{\pi/6} = \frac{2}{3}$
3. $\displaystyle\int_0^{\pi} \left(5\cos 2x - e^{5x} + 3\right)\,dx = \left[\frac{5}{2}\sin 2x - \frac{1}{5}e^{5x} + 3x\right]_0^{\pi} = 3\pi - \frac{1}{5}e^{5\pi} + \frac{1}{5}$
4. $\displaystyle\int_0^{\pi/2} \sin^2 t\cos t\,dt = \left[\frac{\sin^3 t}{3}\right]_0^{\pi/2} = \frac{1}{3}$
5. $\displaystyle\int_0^1 \left(2x + e^{2x} + e^{-x}\right)\,dx = \left[x^2 + \frac{1}{2}e^{2x} - e^{-x}\right]_0^1 = \frac{1}{2}e^2 - e^{-1} + \frac{3}{2}$
6. $\displaystyle\int_1^e \frac{\ln t}{t}\,dt = \int_1^e \frac{1}{t}\ln t\,dt = \left[\frac{\ln^2 t}{2}\right]_1^e = \frac{1}{2}$
7. $\displaystyle\int_{\pi/3}^{\pi/4} \tan t\,dt = \int_{\pi/3}^{\pi/4} \frac{\sin t}{\cos t}\,dt = [-\ln|\cos t|]_{\pi/3}^{\pi/4} = -\ln\frac{\sqrt{2}}{2} + \ln\frac{1}{2} = -\ln\sqrt{2}$
8. $\displaystyle\int_0^3 \frac{1}{\sqrt{x+1}}\,dx = \int_0^3 (x+1)^{-\frac{1}{2}}\,dx = \left[\frac{(x+1)^{\frac{1}{2}}}{\frac{1}{2}}\right]_0^3 = \left[2\sqrt{x+1}\right]_0^3 = 2$
9. $\displaystyle\int_0^1 \frac{e^x}{e^x+1}\,dx = [\ln|e^x+1|]_0^1 = \ln(e+1) - \ln 2 = \ln\frac{e+1}{2}$
10. $\displaystyle\int_1^e \frac{\ln^2 x}{x}\,dx = \int_1^e \frac{1}{x}\ln^2 x\,dx = \left[\frac{\ln^3 x}{3}\right]_1^e = \frac{1}{3}$
11. $\displaystyle\int_2^4 \left(x - \frac{1}{2}\right)\left(x^2 - x + 3\right)^3\,dx = \left[\frac{1}{2}\frac{\left(x^2-x+3\right)^4}{4}\right]_2^4 = 6250$
12. $\displaystyle\int_0^1 2x\sqrt{1+x^2}\,dx = \int_0^1 2x\left(1+x^2\right)^{\frac{1}{2}}\,dx = \left[\frac{\left(1+x^2\right)^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^1 = \frac{4}{3}\sqrt{2} - \frac{2}{3}$

13. $\displaystyle\int_0^1 \frac{2x+1}{\left(x^2+x+1\right)^2}\ dx = \int_0^1 (2x+1)\left(x^2+x+1\right)^{-2}\ dx = \left[\frac{\left(x^2+x+1\right)^{-1}}{-1}\right]_0^1 = \frac{2}{3}$

14. $\displaystyle\int_1^4 \frac{x^3+1}{\sqrt{x}}\ dx = \int_1^4 \left(\frac{x^3}{\sqrt{x}}+\frac{1}{\sqrt{x}}\right)\ dx = \int_1^4 \left(x^{\frac{5}{2}}+x^{-\frac{1}{2}}\right)\ dx = \left[\frac{x^{\frac{7}{2}}}{\frac{7}{2}}+\frac{x^{\frac{1}{2}}}{\frac{1}{2}}\right]_1^4 = \frac{268}{7}$

15. $\displaystyle\int_0^{\ln 3} \frac{e^x-e^{-x}}{e^x+e^{-x}}\ dx = \left[\ln\left|e^x+e^{-x}\right|\right]_0^{\ln 3} = \ln\frac{10}{3} - \ln 2 = \ln\frac{5}{3}$

16. $\displaystyle\int_1^e \frac{(1+\ln x)^2}{x}\ dx = \int_1^e \frac{1}{x}(1+\ln x)^2\ dx = \left[\frac{(1+\ln x)^3}{3}\right]_1^e = \frac{7}{3}$

17. $\displaystyle\int_0^1 e^x\sqrt{1+e^x}\ dx = \int_0^1 e^x(1+e^x)^{\frac{1}{2}}\ dx = \left[\frac{(1+e^x)^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^1 = \frac{2}{3}(1+e)\sqrt{1+e} - \frac{4}{3}\sqrt{2}$

Solution de l'exercice 4

1. $\displaystyle\int_1^e x\ln x\ dx = \frac{1}{4}e^2+\frac{1}{4}$ Poser $u=\ln x$ et $v'=x$
2. $\displaystyle\int_1^2 x^3\ln 4x\ dx = 4\ln 8 - \frac{1}{4}\ln 4 - \frac{15}{16}$ Poser $u=\ln 4x$ et $v'=x^3$
3. $\displaystyle\int_0^1 (x+1)\,e^{-2x}\ dx = \frac{3}{4}-\frac{5}{4}e^{-2}$ Poser $u=x+1$ et $v'=e^{-2x}$
4. $\displaystyle\int_0^1 x\ e^x\ dx = 1$ Poser $u=x$ et $v'=e^x$
5. $\displaystyle\int_0^1 \frac{x}{e^x}\ dx = 1-2e^{-1}$ Poser $u=x$ et $v'=e^{-x}$
6. $\displaystyle\int_1^e \ln x\ dx = 1$ Poser $u=\ln x$ et $v'=1$
7. $\displaystyle\int_1^2 \left(x^2+1\right)\ln 2x\ dx = \frac{14}{3}\ln 4 - \frac{4}{3}\ln 2 - \frac{16}{9}$ Poser $u=\ln 2x$ et $v'=x^2+1$
8. $\displaystyle\int_0^\pi x\cos x\ dx = -2$ Poser $u=x$ et $v'=\cos x$
9. $\displaystyle\int_0^\pi 2x\sin 3x\ dx = \frac{2}{3}\pi$ Poser $u=2x$ et $v'=\sin 3x$
10. $\displaystyle\int_0^3 \frac{x}{\sqrt{x+1}}\ dx = \frac{8}{3}$ Poser $u=x$ et $v'=\frac{1}{\sqrt{x+1}}=(x+1)^{-\frac{1}{2}}$
11. $\displaystyle\int_0^1 xe^{kx}\ dx$ Poser $u=x$ et $v'=e^{kx}$
12. $\displaystyle\int_0^1 \ln(1+t)\ dt = 2\ln 2 - 1$ Poser $u=\ln(1+t)$ et $v'=1$
13. $\displaystyle\int_0^1 \arctan x\ dx = \frac{\pi}{4}-\frac{1}{2}\ln 2$ Poser $u=\arctan x$ et $v'=1$
14. $\displaystyle\int_0^1 x\arctan x\ dx = \frac{\pi}{4}-\frac{1}{2}$ Poser $u=\arctan x$ et $v'=x$
15. $\displaystyle\int_0^{1/2} \arcsin x\ dx = \frac{\pi}{12}+\frac{\sqrt{3}}{2}-1$ Poser $u=\arcsin x$ et $v'=1$

16. $I=\int_{\pi/6}^{\pi/3}\frac{dt}{\cos^2 t\sin t}$ Poser $u=\frac{1}{\sin t}$ et $v'=\frac{1}{\cos^2 t}$ d'où $u'=\frac{-\cos t}{\sin^2 t}$ et $v=\tan t$

    D'où $I=\left[\frac{1}{\cos t}\right]_{\pi/6}^{\pi/3}-\int_{\pi/6}^{\pi/3}\frac{-1}{\sin t}\,dt=2-\frac{2}{\sqrt{3}}-\left[-\ln\left|\tan\frac{t}{2}\right|\right]_{\pi/6}^{\pi/3}=2-\frac{2}{\sqrt{3}}+\ln\tan\frac{\pi}{6}-\ln\tan\frac{\pi}{12}$

    $\ln\tan\frac{\pi}{12}=\ln\frac{\sin\frac{\pi}{12}}{\cos\frac{\pi}{12}}=\ln\sin\frac{\pi}{12}-\ln\cos\frac{\pi}{12}=\ln\sin\left(\frac{\pi}{3}-\frac{\pi}{4}\right)-\ln\cos\left(\frac{\pi}{3}-\frac{\pi}{4}\right)$

    $=\ln\left(\sin\frac{\pi}{3}\cos\frac{\pi}{4}-\sin\frac{\pi}{4}\cos\frac{\pi}{3}\right)-\ln\left(\cos\frac{\pi}{3}\cos\frac{\pi}{4}+\sin\frac{\pi}{4}\sin\frac{\pi}{3}\right)$

    $=\ln\left(\frac{\sqrt{2}}{4}\left(\sqrt{3}-1\right)\right)-\ln\left(\frac{\sqrt{2}}{4}\left(1+\sqrt{3}\right)\right)=\ln\frac{\sqrt{3}-1}{1+\sqrt{3}}=\ln\left(2-\sqrt{3}\right)$

    Finalement : $I=2-\frac{2}{\sqrt{3}}+\ln\frac{1}{\sqrt{3}}-\ln\left(2-\sqrt{3}\right)$

Solution de l'exercice 5

1. $\int_0^1 t^2e^t\,dt=e-2$ Poser $u=t^2$ et $v'=e^t$ puis $u=2t$ et $v'=e^t$
2. $\int_{-\pi}^{\pi}x^2\cos x\,dx=-4\pi$ Poser $u=x^2$ et $v'=\cos x$ puis $u=2x$ et $v'=\sin x$
3. $I=\int_0^{\pi}e^x\sin x\,dx$ Poser $u=e^x$ et $v'=\sin x$

   $I=\left[-e^x\cos x\right]_0^{\pi}-\int_0^{\pi}\left(-e^x\cos x\right)\,dx=e^{\pi}+1+\int_0^{\pi}e^x\cos x\,dx=e^{\pi}+1+J$

   Pour calculer $J=\int_0^{\pi}e^x\cos x\,dx$, poser $u=e^x$ et $v'=\cos x$

   On obtient : $J=\left[e^x\sin x\right]_0^{\pi}-\int_0^{\pi}e^x\sin x\,dx=-I$

   Finalement : $I=e^{\pi}+1+J=e^{\pi}+1-I\Rightarrow 2I=e^{\pi}+1\Rightarrow I=\frac{e^{\pi}+1}{2}$
4. $\int_0^{\pi}e^{-2x}\cos 3x\,dx=e^{-2\pi}\left(\frac{2}{13}e^{2\pi}+\frac{2}{13}\right)$

   Exercice analogue au précédent.

Solution de l'exercice 6

La primitive cherchée est $F(x)=\int_0^x\arcsin t\,dt$

Poser $u=\arcsin t$ et $v'=1$ pour arriver à $F(x)=x\arcsin x+\sqrt{1-x^2}-1$

Solution de l'exercice 7

1. $\int_0^{\ln\sqrt{3}}\frac{e^t}{1+e^{2t}}\,dt=\int_1^{\sqrt{3}}\frac{dx}{1+x^2}=\frac{1}{12}\pi$
2. $\int_0^{\pi/2}\cos^3 x\,dx=\int_0^{\pi/2}\cos^2 x\cos x\,dx=\int_0^{\pi/2}\left(1-\sin^2 x\right)\cos x\,dx=\int_0^1\left(1-t^2\right)\,dt=\frac{2}{3}$
3. $\int_0^1\frac{dx}{1+x+x^2}=\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}}\frac{\frac{\sqrt{3}}{2}dt}{\frac{3}{4}+\frac{3}{4}t^2}=\frac{\pi\sqrt{3}}{9}$
4. $\int_0^8\frac{dx}{1+\sqrt{1+x}}=\int_1^3\frac{2t\,dt}{1+\sqrt{t^2}}=\int_1^3\frac{2t}{1+t}\,dt=\int_1^3\left(2-\frac{2}{t+1}\right)\,dt=2\ln 2-2\ln 4+4=4-2\ln 2$
5. $\int_0^{1/2}\frac{3}{1+4t^2}\,dt=\int_0^1\frac{3}{1+x^2}\frac{1}{2}\,dx=\frac{3}{8}\pi$
6. $\int_0^{1/4}\frac{1}{\sqrt{1-4x^2}}\,dx=\int_0^{\frac{1}{2}}\frac{1}{\sqrt{1-t^2}}\frac{1}{2}\,dt=\frac{\pi}{12}$

7. $\displaystyle\int_0^1 \sqrt{1-t^2}\ dt = \int_0^{\frac{\pi}{2}} (\cos x)\sqrt{\cos^2 x}\, dx$
$\displaystyle = \int_0^{\frac{\pi}{2}} \cos x\,|\cos x|\quad dx = \int_0^{\pi/2} \cos^2 x\ dx = \int_0^{\pi/2} \left(\frac{1}{2}(1+\cos 2x)\right)\ dx = \frac{\pi}{4}$

8. $\displaystyle\frac{\sin 4x}{\sin x} = 4\cos^3 x - 4\cos x \sin^2 x$
Si on pose $t = \sin x$ alors : $x = \arcsin t$ et $\cos x = \cos(\arcsin t) = \sqrt{1-t^2}$ d'où :
$\displaystyle\int_{\pi/4}^{\pi/2} \frac{\sin 4x}{\sin x}\ dx = \int_{\pi/4}^{\pi/2} \left(4\cos^3 x - 4\cos x\sin^2 x\right) dx$
$\displaystyle = \int_{\sqrt{2}/2}^{1} \left(4\left(1-t^2\right)\sqrt{1-t^2} - 4\sqrt{1-t^2}t^2\right)\frac{1}{\sqrt{1-t^2}}\ dt = 4\int_{\sqrt{2}/2}^{1}\left(1-2t^2\right)\ dt = \frac{4}{3} - \frac{4}{3}\sqrt{2}$

9. $\displaystyle\int_0^{\pi/4} \frac{\sin x}{\cos^3 x}\ dx = \int_0^1 t\,dt = \frac{1}{2}$

10. $\displaystyle\int_{-1}^{1} \frac{e^t-1}{e^t+1}\ dt = \int_{e^{-1}}^{e} \frac{x-1}{x+x^2}\, dx = \int_{e^{-1}}^{e}\left(\frac{2}{x+1} - \frac{1}{x}\right)\ dx = \left[2\ln|x+1| - \ln|x|\right]_{e^{-1}}^{e} = 0$
Remarque : le résultat était immédiat à condition d'avoir "vu" que $t \mapsto \dfrac{e^t-1}{e^t+1}$ est impaire.

11. $\displaystyle\int_0^{1/2} \frac{dx}{\sqrt{1-x^2}} = \int_{\frac{1}{2}\pi}^{\frac{1}{3}\pi} (-1)\ dt = \frac{\pi}{6}$
Remarque : on pouvait intégrer immédiatement : $\displaystyle\int_0^{1/2} \frac{dx}{\sqrt{1-x^2}} = [\arcsin x]_0^{1/2} = \frac{\pi}{6}$

12. $\displaystyle\int_{3/2}^{7/4} \frac{dx}{\sqrt{-x^2+3x-2}} = \int_0^{1/2} \frac{1}{\sqrt{-\frac{1}{4}t^2+\frac{1}{4}}}\frac{1}{2}dt = \int_0^{1/2}\frac{1}{\sqrt{1-t^2}}\ dt = [\arcsin x]_0^{1/2} = \frac{\pi}{6}$

13. $\displaystyle\int_0^1 \frac{e^x}{1+e^{-x}}\ dx = \int_1^e \frac{1}{\frac{1}{t}+1}\, dt = \int_1^e \frac{t}{1+t}\ dt = \int_1^e\left(1 - \frac{1}{1+t}\right)\ dt = e - \ln(e+1) - 1 + \ln 2$

14. $\displaystyle\int_0^1 \frac{\arctan x}{1+x^2}\ dx = \int_0^{\frac{1}{4}\pi} t\,dt = \frac{\pi^2}{32}$

15. $\displaystyle\int_0^1 \frac{1}{\sqrt{3-2x}}\ dx = \int_3^1 \frac{1}{\sqrt{t}}\frac{-1}{2}dt = \sqrt{3} - 1$

16. $\displaystyle\int_2^{7/2} \frac{dt}{\sqrt{-t^2+4t+5}} = \int_0^{\frac{1}{2}} \frac{3}{\sqrt{9-9x^2}}\, dx = \int_0^{1/2}\frac{dx}{\sqrt{1-x^2}} = [\arcsin x]_0^{1/2} = \frac{\pi}{6}$

Solution de l'exercice 8

1. $\displaystyle\int_2^3 \frac{1}{1-t^2}\ dt = \int_2^3 \left(\frac{1}{2(t+1)} + \frac{1}{2(1-t)}\right)\ dt = \frac{1}{2}\ln 4 - \frac{1}{2}\ln 3 - \frac{1}{2}\ln 2$

2. $\displaystyle\int_{-2}^{-1} \frac{t-2}{t(t-1)}\ dt = \int_{-2}^{-1}\left(\frac{2}{t} - \frac{1}{t-1}\right)\ dt = \ln 3 - 3\ln 2$

3. $\displaystyle\int_{-1}^{0} \frac{t+1}{t^2+t-2}\ dt = \int_{-1}^{0}\left(\frac{2}{3(t-1)} + \frac{1}{3(t+2)}\right)\ dt = -\frac{1}{3}\ln 2$

4. $\displaystyle\int_{-4}^{-2} \frac{t+1}{t^2+4t+8}\ dt$ Poser $\displaystyle x = \frac{2t+4}{\sqrt{16}} = \frac{1}{2}t + 1$
$\displaystyle\int_{-1}^{0} \frac{2x-1}{2x^2+2}\, dx = \int_{-1}^{0}\left(\frac{1}{2}\frac{2x}{x^2+1} - \frac{1}{2}\frac{1}{x^2+1}\right)\ dx = \left[\frac{1}{2}\ln\left|x^2+1\right| - \frac{1}{2}\arctan x\right]_{-1}^{0} = -\frac{1}{2}\ln 2 - \frac{1}{8}\pi$

5. $\displaystyle\int_2^{e+1} \frac{t+1}{t-1}\ dt = \int_2^{e+1}\left(1 + \frac{2}{t-1}\right)\ dt = [t + 2\ln|t-1|]_2^{e+1} = e + 1$

6. $\displaystyle\int_3^4 \frac{t^3+1}{t(t-2)}\,dt = \int_3^4 \left(t+2-\frac{1}{2t}+\frac{9}{2}\frac{1}{t-2}\right)\,dt = \frac{9}{2}\ln 2+\frac{1}{2}\ln 3-\frac{1}{2}\ln 4+\frac{11}{2}$

7. $\displaystyle\int_1^2 \frac{t^3+2t^2+4t+2}{t^2+t}\,dt = \int_1^2 \left(t+1+\frac{2}{t}+\frac{1}{t+1}\right)\,dt = \ln 2+\ln 3+\frac{5}{2}$

8. $\displaystyle\int_{1/3}^{2/3} \frac{7t^2-2t-1}{t+2t^2-3t^3}\,dt = \int_{1/3}^{2/3} \left(\frac{1}{1-t}-\frac{1}{t}-\frac{1}{3t+1}\right)\,dt = \frac{1}{3}\ln 2-\frac{1}{3}\ln 3$

9. $\displaystyle\int_{1/2}^{4/3} \frac{6t^2+31t+12}{6t^2+19t-7}\,dt = \int_{1/2}^{4/3} \left(\frac{3}{3t-1}+\frac{2}{2t+7}+1\right)\,dt = \ln 3-\ln 2-\ln 6+\ln 29+\frac{5}{6}$

Solution de l'exercice 9

1. $\displaystyle\int_{-1}^1 \frac{x-1}{x^2-4}\,dx = \frac{1}{2}\ln 3$

2. $\displaystyle\int_0^1 \frac{dx}{(x+1)(x+2)} = 2\ln 2-\ln 3$

3. $\displaystyle\int_{-2}^0 \frac{5}{x^2+4x+8}\,dx = \frac{5}{8}\pi$

4. $\displaystyle\int_0^2 \frac{x^2+x-1}{x^2-2x-3}\,dx = 2-\frac{5}{2}\ln 3$

5. $\displaystyle\int_{-2}^{-1} \frac{x}{x^2+4x+5}\,dx = \frac{1}{2}\ln 2-\frac{1}{2}\pi$

6. $\displaystyle\int_0^1 \frac{dx}{(x+1)(x+2)(x+3)} = \ln 2-\frac{3}{2}\ln 3+\frac{1}{2}\ln 8$

7. $\displaystyle\int_3^4 \frac{x^4+x+1}{x^3-3x^2+2x}\,dx = \frac{25}{2}\ln 2-\frac{7}{2}\ln 3+\frac{1}{2}\ln 4+\frac{13}{2}$

Solution de l'exercice 10 $\displaystyle\frac{1}{x(x^2+1)} = \frac{1}{x}-\frac{x}{x^2+1}$

$$\int_1^2 \frac{x^3+x+1}{x(x^2+1)}\,dx = \int_1^2 \left(\frac{1}{x}-\frac{x}{x^2+1}\right)\,dx = \frac{3}{2}\ln 2-\frac{1}{2}\ln 5$$

Solution de l'exercice 11 $\displaystyle\frac{1}{x^4-1} = \frac{1}{4(x-1)}-\frac{1}{4(x+1)}-\frac{1}{2(x^2+1)}$

$$\int_0^{\sqrt{3}/3} \frac{x^4}{x^4-1}\,dx = \int_0^{\sqrt{3}/3} \left(\frac{1}{4(x-1)}-\frac{1}{4(x+1)}-\frac{1}{2(x^2+1)}\right)\,dx$$
$$= \left[\frac{1}{4}\ln|x-1|-\frac{1}{4}\ln|x+1|-\frac{1}{2}\arctan x\right]_0^{\sqrt{3}/3} = \frac{1}{4}\ln\left(1-\frac{\sqrt{3}}{3}\right)-\frac{1}{4}\ln\left(1+\frac{\sqrt{3}}{3}\right)-\frac{\pi}{12}$$

Solution de l'exercice 12 $\displaystyle\frac{2x+1}{(x+1)^2} = \frac{2}{x+1}-\frac{1}{(x+1)^2}$

$$\int_0^1 \frac{2x+1}{(x+1)^2}\,dx = \int_0^1 \left(\frac{2}{x+1}-\frac{1}{(x+1)^2}\right)dx = 2\ln 2-\frac{1}{2}$$

Solution de l'exercice 13 $\displaystyle\int_0^1 e^{-x}\ln(1+e^x)\,dx = \int_1^e \frac{1}{t^2}\ln(t+1)\,dt$

$$= \ln 2-\frac{1}{e}\ln(e+1)-\int_1^e\left(-\frac{1}{t(t+1)}\right)\,dt = 2\ln 2-\ln(e+1)-\frac{1}{e}\ln(e+1)+1$$

Solution de l'exercice 14

1. $I = \int_1^e \frac{\ln x}{x}\ dx = \frac{1}{2}$

2. Poser $\varphi(x) = \frac{x}{1+x^2} - \frac{1}{2x}$ et $\psi(x) = \frac{1}{x} - \frac{1}{1+x^2}$
   $\varphi(x) = \frac{(x+1)(x-1)}{2x(1+x^2)}$ et $\psi(x) = \frac{x^2-x+1}{x(1+x^2)}$
   $\forall x \in [1;+\infty[: \varphi(x) \geq 0$ et $\psi(x) > 0$ (un trinôme ax^2+bx+c dont le discriminant est strictement négatif a toujours le signe de a)
   D'où $\forall x \in [1;+\infty[: \frac{1}{2x} \leq \frac{x}{1+x^2} < \frac{1}{x}$ (résultat plus fort que la question posée)

3. De 2. on déduit : $\forall x \in [1;+\infty[: \frac{\ln x}{2x} \leq \frac{x\ln x}{1+x^2} < \frac{\ln x}{x}$ car $\ln x \geq 0$ sur $[1;+\infty[$
   D'où : $\int_1^e \frac{\ln x}{2x}\ dx \leq \int \frac{x\ln x}{1+x^2}\ dx \leq \int_1^e \frac{\ln x}{x}\ dx$ ce qui donne : En déduire un encadrement de l'intégrale $J = \int_1^e \frac{x\ln x}{1+x^2}\ dx$
   Or $\int_1^e \frac{\ln x}{2x}\ dx = \frac{1}{4}$ et $\int_1^e \frac{\ln x}{x} = 2\int_1^e \frac{\ln x}{2x}\ dx = \frac{1}{2}$
   Donc $\frac{1}{4} \leq J \leq \frac{1}{2}$

Solution de l'exercice 15 $I = \int_0^{\pi/2} \frac{\cos x}{1+2\sin x}\ dx$ et $J = \int_0^{\pi/2} \frac{\sin 2x}{1+2\sin x}\ dx$

1. $I = \int_0^{\pi/2} \frac{\cos x}{1+2\sin x}\ dx = \left[\frac{1}{2}\ln|1+2\sin x|\right]_0^{\pi/2} = \frac{1}{2}\ln 3$
   $I + J = \int_0^{\pi/2} \frac{\cos x + \sin 2x}{1+2\sin x}\ dx = \int_0^{\pi/2} \frac{\cos x + 2\sin x\cos x}{1+2\sin x}\ dx = \int_0^{\pi/2} \cos x\ dx = 1$

2. $J = I + J - I = 1 - \frac{1}{2}\ln 3$